**Propagating Uncertainty in calculations**

Often in labs we’ll produce secondary data by doing calculations using the raw data values we have measured. Since the raw data numbers we use to do the calculations have uncertainty associated with them, the result of the calculation (the secondary data) must also have uncertainty. When you do calculations with values that have uncertainties, you will need to determine the uncertainty in the result. How uncertainty of a calculated result is affected by the uncertainty of the numbers used to make the calculation is called ***uncertainty propagation***.

A simple way to estimate uncertainties this is to find the *largest possible uncertainty* the calculation could yield. This will always overestimate the uncertainty of your calculation, but an overestimate is considered better than an underestimate.

The following sections, we’ll learn how to express the uncertainty of the results of calculations made using numbers with uncertainty.

# Addition and Subtraction

Suppose we measure the length of two tables using a meter stick. For the sake of this example, let’s say that our measurements have uncertainty of ±1 cm. Table 1 measures 165 ± 1 cm long and Table 2 measures 155 ± 1 cm long. Now suppose we put the two tables end to end. How long do we think the tables will be together?

Based on the minimum and maximum lengths including the uncertainties, the two tables combined length could be any value between 318 cm and 322 cm.

##  Length of two tables combined = 320 ± 2cm

Do you see that there is an easier way to reach the same answer? If we added the original measurements, we’d get the midpoint, 320 cm in this case. If we add the absolute uncertainties of the numbers we are adding, we’d get the combined absolute uncertainty, 2 cm in this case. We can generalize this as a rule we can follow for propagating uncertainties when adding or subtraction numbers which have uncertainty:

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| --- | --- |
| ***Sum and Difference Rule:***  | *When adding or subtracting values with uncertainty, add absolute uncertainties of each value to find the absolute uncertainty of the result.*  |

**ADDING and SUBTRACTING rule for uncertainties:**

When adding or subtracting values with uncertainty, add absolute uncertainties of each value to find the absolute uncertainty of the result.

Mathematically, this looks like:

ADDING x and y Subtracting y from x:

(x ± Δx) + (y ± Δy) = x + y ± (Δx + Δy ) OR (x ± Δx) - (y ± Δy) = x - y ± (Δx + Δy )

# Multiplication and Division

Now suppose we multiply two measured values with uncertainty. For example, if we measured the width of a table as 55 ±1 cm, in addition to the length of 165±1 cm. To calculate the area of the table top we would multiply these two values. But what is the uncertainty of this area given both width and the length have uncertainty?

This process for determining uncertainty when two values are multiplied or divided is more complicated than when adding and subtracting values. The rule can be simply stated as

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| ***Multiplying and dividing Rule:***  | *When multiplying or dividing values with uncertainty, add the relative uncertainties of each value to find the relative uncertainty of the result. Then convert to an absolute uncertainty.* |

Multiplying x and y: Dividing x by y:

 (x ± Δx) × (y ± Δy) = x × y ± (x × y )(Δx/x + Δy/y) (x ± Δx) ÷ (y ± Δy) = x ÷ y ± (x ÷ y )(Δx/x + Δy/y)

# There is a longer way of doing this than using the complex formulas above.

Step 1 – convert the absolute uncertainties of both x and y to percent uncertainties of x and y

Step 2 – add the percent uncertainties of x and y to get a total percent uncertainty

Step 3 – calculate the answer to your calculation, multiplication of division.

Step 4 – covert the total percent uncertainty back to absolute uncertainty by multiplying by the answer in step 3, and dividing by 100.

Step 5 – write the answer (step 3) and the absolute uncertainty (step 4) together using the ± symbol.

# Significant Figures when expressing absolute uncertainty

Sometimes calculation of two relatively simple numbers creates answers with a lot of significant figures.

For example: consider the calculation of dividing the area of a desk by its length, in order to calculate the width of the desk.

Area (± uncertainty) ÷ length (± uncertainty) = width (± uncertainty,calculated using formula above)

 2200 ± 40 cm2 ÷ 59.0 ± 1.6 cm = 37.2881 ± 1.6892 cm

It is clear that the added precision implied by all these decimal places is not real. Remember that our uncertainty in the original measurements was only one decimal place. We can’t have uncertainty in our result to 4 decimal places. This implies accuraty to $\frac{1}{10000}$cm – it is meaningless to use so many decimals.

As a general rule, we limit absolute uncertainty to a maximum of one decimal place more than the measurement itself. For the example above, we’d express the answer to no decimal places (two significant figures), and the uncertainty to one decimal place.

Width = 37 ± 1.7 cm

Again, there are other rules for writing the uncertainty. An alternative often used is to write the final uncertainty to only one significant figure. There no “right” way, so your teacher will identify the preferred method for your subject. The methods presented here are easy to follow – so that is what we will use at Maroochydore SHS (blame Mr Turner If you get in trouble!). Most importantly, they show that we must continue to consider the uncertainty of a measurement when using the measurement for calculations.

Finally, remember that these techniques help you estimate the uncertainty that always occurs in measurements. They will not help account for a poor method, mistakes, or poor measurement – all systemic errors. Careful and thoughtful measurements are essential to produce quality data.

# Problems

1. *Students measuring the dimensions of a table top use a meter stick. They determine that the width of the table is between 78.4 cm and 78.3 cm.*
2. *What is the absolute uncertainty of the width measurement?*
3. *What is the percent (relative) uncertainty of the width measurement?*
4. *Using the same meter stick to measure the thickness of the table, the students determine that the thickness is between 3.5 cm and 3.6 cm.*
5. *What is the absolute uncertainty of the thickness measurement?*
6. *What is the percent (relative) uncertainty of the thickness measurement?*
7. *Compare the percent (relative) uncertainties of the width and thickness. Why are they so different if the same meter stick was used for each measurement?*
8. *Consider the following results for different experiments. Determine if they agree with the accepted or predicted result listed to the right.*
	1. *measured value for g = 10.4 ± 1.1 m/s2 (accepted value for g = 9.8 m/s2)*
	2. *measured value for T = 1.5 ± 0.1 sec (predicted value for T = 1.1 sec)*
	3. *measured value for k = 1368 ± 45 N/m (predicted value for k = 1300 ± 50 N/m)*
9. *Each member of your lab group weighs an empty box and two metal bars twice. The following table shows this data.*

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| --- | --- | --- | --- | --- | --- | --- |
| *trial*  | ***Box (g)***  | *Random Uncertainty*  | ***Bar 1 (g)***  | *Random Uncertainty*  | ***Bar 2 (g)***  | *Random Uncertainty*  |
| *1*  | *201.3*  |  | *98.7*  |  | *95.6*  |  |
| *2*  | *201.5*  | *98.8*  | *95.3*  |
| *3*  | *202.3*  | *96.9*  | *96.4*  |
| *4*  | *202.1*  | *97.1*  | *96.2*  |
| *5*  | *199.8*  | *98.4*  | *95.8*  |
| *6*  | *200.0*  | *98.6*  | *95.6*  |
| *average*  |  | *±*  |  | *±*  |  | *±*  |

* 1. *Estimate the random uncertainty of each data set.*
	2. *Calculate the total mass of the box with Bar 1 inside the box. Use rules for uncertainty propagation.*
	3. *Calculate the mass of the box with Bar 2 inside the box. Use rules for uncertainty propagation.*
	4. *Calculate the mass of the box with both bars inside the box. Use rules for uncertainty propagation.*
1. *The area of a rectangular metal plate was found by measuring its length and its width. The length was found to be 5.37±0.05 cm. The width was found to be 3.42±0.02 cm.*
	1. *What are the percent (relative) uncertainties of each measurement?*
	2. *What is the area, including the uncertainty? (propagate the uncertainties.)*

# Discussion Questions

1. *What is the difference between uncertainty and error?*
2. *Students just starting science often attribute results that they think are incorrect to “human error”. More advanced science students recognize that this is not a sufficient description of potential problems in lab work.* *Why?*
3. *You are measuring the time it takes for a student to run a 100-meter race. Describe a method you could use to determine the measurement uncertainty of the time and the random uncertainty in the time.*
4. *What things can we be absolutely certain about?*

# Sample Quiz Questions

1. *Students are trying to identify an unknown liquid by determining its density and comparing it to a table of densities of known liquids. They begin by finding the mass of a graduated cylinder, which they determine to be 54.55 ± 0.05 grams. What is the percent (relative) uncertainty of this measurement?*
2. *The scale at right shows the mass of the graduated cylinder from problem 2 filled with some of the unknown liquid. Determine the reading on the beam balance at right, including absolute uncertainty. What is the percent (relative) uncertainty of the measurement?*
3. *What is the mass of the liquid in the graduated cylinder, including uncertainty? What is the percent (relative) uncertainty of this measurement?*
4. *By reading the graduated cylinder, the students determine that the volume of liquid is 114 ± 2 ml. What is the density of the unknown liquid, including uncertainty? (note: use the method of adding percent (relative) uncertainties)*

|  |  |
| --- | --- |
| *Compound*  | *Density (g/ml)*  |
| *Methanol*  | *0.791*  |
| *Ethanol*  | *0.789*  |
| *Isopropanol*  | *0.785*  |

1. *Shown at right is a table of densities of various alcohols. What conclusions can the students reach about the identity of the unknown liquid based on this table and the results of their density calculations?*
2. *Identify one plausible source of a systematic error made in this procedure and describe how to correct it.*
3. *Identify one source of a random error made in this procedure and describe how to correct it.*